Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON CERTAIN REAL SOLUTIONS OF BABBAGE'S FUNCTIONAL EQUATION.*

By J. F. Ritt.

1. The first attempt to find the most general solution of the functional equation

$$f^n(x) = x,$$

where $f^2(x) = f[f(x)]$, and in general, $f^n(x) = f[f^{n-1}(x)]$, was made by C. Babbage, who published his results in Herschel's "Calculus of Finite Differences" in 1820. Babbage observed that if $f(x)$ is a particular solution of this equation, the formula

$$F(x) = \varphi^{-1}f\varphi(x),$$

$\varphi(x)$ being an arbitrary function, and $\varphi^{-1}(x)$ its inverse, will give an infinite number of other solutions. To verify this one need but calculate $F^n(x)$. For the particular solution $f(x)$ one can use, for instance, the well known "periodic transformation"

$$f(x) = \frac{\alpha + \beta x}{\gamma + \delta x}$$

where

$$\delta = -\frac{\beta^2 - 2\beta\gamma\cos\frac{2k\pi}{n} + \gamma^2}{2\alpha\left(1 + \cos\frac{2k\pi}{n}\right)},$$

k being any integer prime to n.†

To avoid the very ambiguous term "periodic" we shall refer briefly to a solution of Babbage's equation as a "function of order n," understanding therein that n is the lowest order of the function. Also, we may occasionally say "$f(x)$ circulates with a period n."

Babbage called $F(x) = \varphi^{-1}f\varphi(x)$ the general solution of his equation, but, as Pincherle explains,‡ the term "general" refers only to the presence of the arbitrary function $\varphi(x)$ and does not signify complete generality. In fact, Babbage§ himself gave an example to show that his solution lacked generality, but his conclusions have little value from the point of view of the modern theory of functions.

---

* Read under a different title before the American Mathematical Society, February 27, 1915.

† Boole, "Calculus of Finite Differences," p. 208.

‡ Encyclopédie des Sciences Mathématiques, II, 26.

§ Philosophical Transactions, 1817, p. 206.

Babbage's equation was also studied by Leau,[*] who confined himself to the analytic solutions.[†]

The object of this paper is to study a certain class of real solutions of Babbage's equation, and to determine whether they are all contained in the formula $\varphi^{-1}f\varphi(x)$. We shall, with certain assumptions as to $\varphi(x)$, $\varphi^{-1}(x)$ and the solutions of the equation, find the most general solution to be, when $n > 2$,

$$F(x) = \varphi^{-1}f^p\varphi(x)$$

where p takes on all integral values prime to n. It will be necessary to consider separately the case of $n = 2$.

2. It is evident that $\varphi^{-1}f\varphi(x)$ will be the most general solution if, when given two solutions, $F(x)$ and $f(x)$, we can find a function $\varphi(x)$ such that

$$F(x) = \varphi^{-1}f\varphi(x).\ddagger$$

We shall try to construct such a function $\varphi(x)$. Suppose $\varphi(a) = b$. Observing that if $\varphi(x)$ exists, $F^p(x) = \varphi^{-1}f^p\varphi(x)$, we must have

$$\varphi F^p(a) = f^p\varphi(a) = f^p(b).$$

In particular, $\varphi F^n(a) = f^n(b)$, and this condition reduces to $\varphi(a) = b$, as assumed. Conversely, if we define $\varphi(x)$ as indicated for the n points $x = F^p(a)$, $(p = 0, 1, 2, \cdots, n-1)$, we shall have, for those points, $F(x) = \varphi^{-1}f\varphi(x)$. Similarly, if we put $\varphi(a') = b'$, we can define $\varphi(x)$ for n more points. If some of the points $F^p(a)$, $F^p(a')$ coincide, it may be necessary to take $\varphi(x)$ as a many-valued function.

Thus we can proceed, defining $\varphi(x)$ for an indefinite number of points, and the generality of Babbage's solution appears quite complete. However, if we propose to define $\varphi(x)$ for the entire domain of x and to make it one-valued and generally continuous, the problem is far from being settled. It might be proposed to define $\varphi(x)$ arbitrarily in some interval (a, a'), say as some continuous function $\Phi(x)$, then to define it as $f\Phi F^{-1}(x)$ in the interval $[F(a), F(a')]$, as $f^2\Phi F^{-2}(x)$ in $[F^2(a), F^2(a')]$, etc., according to the principle explained above. But one might object, for instance, that however small the interval (a, a') be taken, $F(x)$ will lie in (a, a') for some values of x in (a, a'), and that it will then be out of the question to define $\varphi(x)$ arbitrarily.

Thus the problem will be solved if we can divide the domain of x into n intervals, such that if x lies in the first interval, $f(x)$ will lie in the second,

---

[*] Bulletin de la Société Mathématique de France, 1898.

[†] See also A. A. Bennet, Annals of Mathematics, Vol. 17, 1915, p. 37.

[‡] This principle was used by Babbage, *Phil. Trans.*, 1817.

$f^2(x)$ in the third, etc. It will then be possible to take $\varphi(x)$ arbitrarily in one of these intervals, and its behavior in the entire domain of x will be determined.

3. In order to advance in this direction, we take up the study of a function $f(x)$, of order n, making the following assumptions.

(1) It is defined uniquely for all real values of x, including $+\infty$ and $-\infty$.

(2) It is continuous everywhere, except at an isolated set of points, where it becomes infinite. At these it may approach the same infinity from both sides, in which case it is to be defined as that infinity, or it may approach $+\infty$ from one direction and $-\infty$ from the other, in which case it is to be defined either as $+\infty$ or as $-\infty$. Should $f(x)$ approach a limit, finite or infinite, as x approaches $+\infty$, it is to be defined, for $x = +\infty$, as that limit. If no limit is approached, the definition is to be made in some other manner. Similar remarks apply to $x = -\infty$. If $f(+\infty) = f(-\infty)$, we shall not distinguish between $+\infty$ and $-\infty$.

(3) $f^n(x) = x$ for every value of x, but the integer p not being divisible by n, we can find, in any interval, however small, a point x_1 such that $f^p(x_1) \neq x_1$.

The first two of the above assumptions are to apply also to $\varphi(x)$ and to $\varphi^{-1}(x)$.

4. The function $f^p(x)$, p being any positive integer, takes on every real value for some value of x. For, taking the positive integer k such that $kn > p$,

$$f^p[f^{kn-p}(c)] = f^{kn}(c) = c,$$

where c is any real number, including $\pm\infty$.

Also, the value c is assumed for only one value of x.* For if

$$f^p(\alpha_1) = f^p(\alpha_2) = c,$$

then

$$f^{kn}(\alpha_1) = f^{kn-p}(c) = f^{kn}(\alpha_2)$$

or

$$\alpha_1 = \alpha_2.$$

Thus $f^p(x)$, which is itself one-valued by § 3, has a one-valued inverse, defined for all real values of x, including $+\infty$ and $-\infty$. It follows without difficulty that if we denote the inverse of $f^p(x)$ by $f^{-p}(x)$, we may, without ambiguity, allow the index p to assume all integral values, positive and negative, using the formula $f^p[f^q(x)] = f^{p+q}(x)$ and calling $f^p(x)$ and $f^q(x)$ identical when $p \equiv q$ [mod. n].

5. From the fact that $f(x)$ and its inverse are both one-valued, it follows that $f(x)$ increases or decreases monotonically in any interval in

---

* This fact was first observed by Leau, loc. cit.

which it is continuous. For the same reason, if $f(x)$ becomes discontinuous at a point it must approach $+\infty$ from one direction and $-\infty$ from the other. Finally, if $f(x)$ is discontinuous for $x = c$, it will be bounded, and therefore continuous, in any interval not containing c. In fact $f(x)$ must be bounded in the entire domain of x, if we reject some interval about c. The formal proof of these facts will be found very simple.

6. From §§ 4 and 5, we see that the following are the only possible types of $f(x)$:

Type I. Discontinuous at one point c, and monotone increasing wherever continuous.

As x increases from c to $+\infty$, $f(x)$ increases monotonically to a finite limit a, taking on all values less than a. As x decreases from c to $-\infty$, $f(x)$ decreases monotonically from $+\infty$, and since, by § 4, it must now take on all values greater than a, and these only, it must approach a as x approaches $-\infty$. That is, the graph of $f(x)$ consists of two infinite branches, with the asymptotes $x = c$, and $y = a$.

If $x_1 < x_2$ and c does not lie in the interval (x_1, x_2),

$$f(x_1) < f(x_2),$$

but if c does lie in the interval (x_1, x_2),

$$f(x_1) > f(x_2).$$

Still we shall call $f(x)$ *increasing*.

Type II. Discontinuous at one point c, and monotone decreasing wherever continuous.

Similar remarks apply as to *Type I*.

Type III. Continuous everywhere, and monotone decreasing.

As x increases from $-\infty$ to $+\infty$, $f(x)$ decreases monotonically from $+\infty$ to $-\infty$.

Type IV. Continuous everywhere, and monotone increasing.

As x increases from $-\infty$ to $+\infty$, $f(x)$ increases monotonically from $-\infty$ to $+\infty$.

Since, in the discussion of $f(x)$, we have used the fact that $f^n(x) = x$ only to show that $f^{-1}(x)$ satisfies the first assumption of § 3, it is clear that $\varphi(x)$ and $\varphi^{-1}(x)$ also belong to the above four types.*

---

* Dr. G. M. Green has made the interesting observation that the four types of functions under consideration are all contained in the formula

$$f(x) = \frac{a + bF(x)}{c + dF(x)}$$

where a, b, c and d are real constants such that $ad - bc \neq 0$, and where $F(x)$ is of *Type IV*.

It is not difficult to see that $f(x)$ as thus determined will always be of one of our four types.

7. We leave to the reader the formal work of showing that a function formed by compounding any number of functions of the four types of § 6 will belong to one of those four types, and will be increasing or decreasing according as an even number or an odd number of decreasing functions are employed.

In particular, if $f(x)$ circulates with a period n, every $f^p(x)$ will belong to one of the four types of § 6. If p is even, $f^p(x)$ will be increasing, and if p is odd, $f^p(x)$ will have the same monotonic character as $f(x)$.* Also, if d is the highest common divisor of p and n, then $f^p(x)$ circulates with a period n/d, and m being any integer, not divisible by n/d, we can find in any interval, however small, a point x_1 such that $f^{mp}(x_1) \neq x_1$.

8. Consider a function $f(x)$ of order $n > 1$, and of *Type I*, discontinuous for $x = c$. Suppose $f(a) = a$ for some finite a. Then $f^p(a) = a$ for every p, so that every $f^p(x)$ is finite, and therefore continuous, at a. Thus we can find a δ, where $0 < \delta < | c - a |$, such that $| f^p(x) - a | < | c - a |$ for $| x - a | < \delta$ and for every p. Take a point x_1 in the interval $(a - \delta, a + \delta)$ such that $f(x_1) \neq x_1$. This is possible by the third assumption of § 3. Then c cannot lie between $f^p(x_1)$ and $f^q(x_1)$ for any values of p and q, else either $| f^p(x_1) - a |$ or $| f^q(x_1) - a |$ would be greater than $| c - a |$. Hence, since $f(x)$ is monotone increasing in any interval not containing c,

$$f^{p+1}(x_1) > f^{q+1}(x_1), \text{ if } f^p(x_1) > f^q(x_1).$$

To fix our ideas, suppose $f(x_1) > x_1$. Then

$$x_1 < f(x_1) < f^2(x_1) < \cdots < f^n(x_1) = x_1.$$

Hence we cannot have $f(a) = a$ for any finite a.

Suppose now $f(x)$, of order $n > 1$, is of *Type IV*. Take a point x_1 such that $f(x_1) \neq x_1$; suppose $f(x_1) > x_1$, for instance. Then, as above,

$$x_1 < f(x_1) < f^2(x_1) < \cdots < f^n(x_1) = x_1,$$

from which it is evident that $f(x)$ cannot be of *Type IV* except in the trivial case of $f(x) = x$.

We see now that if $f(x)$, of order $n > 1$, is of *Type I*, $f^p(x)$ will be of *Type I* if p is not divisible by n; for $f^p(x)$ is increasing and, being of order greater than unity, cannot be of *Type IV*. Then we cannot have

---

Also it is evident that the formula gives all functions of *Type III* and *Type IV*. If $f(x)$ is of one of the first two types, it is easy to see that $F(x) = f(x) / [f(x) - f(\infty)]$ is of *Type III* or of *Type IV*. Solving this equation for $f(x)$, we find $f(x)$ exhibited in the desired form. We inserted this note while correcting the proof, but see no way, as yet, for putting this interesting result to profit, principally because it does not lead to a simple expression for $f^p(x)$.

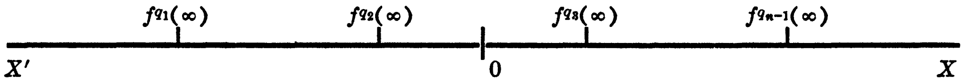
* It follows at once that if n is odd $f(x)$ must be increasing, else $f^n(x) = x$ must decrease as x increases.

$f^p(a) = a$ for any finite a, if p is not divisible by n. Also, if $f^p(\infty) = \infty$, then $f^p(c) = c$, so that the same applies to ∞.

9. The reader will find no difficulty in showing that if $f(x)$ is of *Type II*, we must have $f(x) = x$ for two points, one in the interval $(-\infty, c)$, the other in $(c, +\infty)$. But if $f(x)$ is of order $n > 2$, $f^2(x)$ will be of order greater than unity, and will be of *Type I*. Then since $f^2(x) = x$ when $f(x) = x$, we are led to a contradiction of the results of the preceding article. Hence, *Type II* is impossible for $n > 2$.

If $f(x)$, of order $n > 2$ is of *Type III*, then $f^2(x)$ will be of order greater than unity and of *Type IV*. Hence *Type III* is impossible for $n > 2$.

10. Thus, n being greater than 2, and p not divisible by n, every $f^p(x)$ will be of *Type I* and will be discontinuous for $x = f^{1-p}(c)$. These points of discontinuity are all distinct for $p = 1, 2, \cdots, n-1$, else we would have $f^{q_1}(c) = f^{q_2}(c)$, or $f^{q_1 - q_2}(c) = c$, where $q_1 \neq q_2$. We shall plot these points, designating the point of discontinuity of $f^p(x)$ by $f^{-p}(\infty)$, and using the indices $q_1, q_2, \cdots, q_n$, of which none is divisible by n, and of which no two differ by a multiple of n.



The function $f^{-q_1}(x)$ is continuous and monotone increasing in any interval not containing the point $f^{q_1}(\infty)$. Hence, for values of r from 2 to $n-2$ inclusive, we must have

$$f^{-q_1}[f^{q_{r+1}}(\infty)] > f^{-q_1}[f^{q_r}(\infty)].$$

In virtue of this inequality we must have

$$f^{-q_1}[f^{q_{r+1}}(\infty)] = f^{q_r}(\infty),$$

and consequently

$$q_{r+1} \equiv q_r + q_1 \quad [\text{mod. } n],$$

where, now, r may also be unity.

Therefore, neglecting multiples of n,

$$q_r = rq_1,$$

for all values of r from 1 to $n-1$ inclusive.

Since no two of the indices, $q_1, q_2, \cdots, q_{n-1}$, are congruent to each other with respect to n, the index q_1 must be prime to n. We shall now designate q_1 by the letter m, which we shall call the *scale* of $f(x)$.

It is clear, now, that as x increases from $-\infty$ to $f^m(\infty)$, $f^m(x)$ will increase continuously from $f^m(\infty)$ to $f^{2m}(\infty)$, $f^{2m}(x)$ similarly from $f^{2m}(\infty)$ to $f^{3m}(\infty)$ and finally $f^{(n-1)m}(x)$ will increase monotonically from $f^{(n-1)m}(\infty)$

to $+\infty$. Hence we have accomplished the main purpose of our investigation, for $n > 2$, having divided the domain of x into n intervals such that if x_1 lies in one of them, $f(x_1)$ will lie in a second, etc. If we arrange the numbers $f^p(x_1)$ in order of magnitude (ascending), thus:

$$f^{q_1}(x_1), f^{q_2}(x_1), \cdots, f^{q_n}(x_1),$$

the point $f^{q_p}(x_1)$ will lie between $f^{(p-1)m}(\infty)$ and $f^{pm}(\infty)$. The indices $q_1, q_2, \cdots, q_n$, will form an arithmetic progression whose constant difference is the scale of $f(x)$; that is, neglecting multiples of n.

11. If $f(x)$ is of order n, $f^p(x)$ will have the same order if p and n are relatively prime. Also, if $f(x)$ has the scale m and $f^p(x)$ the scale m',

$$m \equiv pm' \quad [\text{mod. } n].$$

If, then, we can find a function $f(x)$ of order n, we can determine p through the above congruence so that $f^p(x)$ has any desired scale, prime to n. The "periodic transformations" assure us of the existence of functions of all orders, so that all scales prime to n are possible.

12. The reader will find it interesting to verify the results of the foregoing sections in the case of the linear fractional "periodic transformations." For instance, the increasing character of $f(x)$, for $n > 2$, can be shown by differentiating $f(x)$ and using the formula for δ of § 1. Using formulæ for $f^p(x)$ as given by Boole, it is seen that the roots of the equation $f^p(x) = x$ are imaginary unless p is a multiple of n. Also, a little investigation will reveal the scale of $f(x)$. One will find

$$km \equiv \pm 1 \quad [\text{mod. } n],$$

where k is the integer, prime to n, mentioned in § 1, and where the second member has the same sign as

$$\frac{\beta+\gamma}{2\delta}\sin\frac{2k\pi}{n}.$$

13. The argument of § 9 which excludes *Types II* and *III* for $n > 2$, does not apply when $n = 2$. In fact, when $n = 2$, *Types I*, *II* and *III* exist, as the following examples show.

Type I.

$$f(x) = \frac{\alpha - \beta x}{\beta + \delta x}, \qquad \text{where} \qquad \beta^2 + \alpha\delta < 0.$$

Type II.

$$f(x) = \frac{\alpha - \beta x}{\beta + \delta x}, \qquad \text{where} \qquad \beta^2 + \alpha\delta > 0.$$

Another example is $f(x) = \log(1 - e^x)$ in $(-\infty, 0)$ and $f(x) = -\log(1 - e^{-x})$ in $(0, +\infty)$.

Observe that $f(x) = x$ for two real values of x.

Type III.

$$f(x) = a - x.$$

Observe that $f(x) = x$ for a single value of x.

In the case of $n = 2$, the graph of $f(x)$ must be symmetric with respect to the line $y = x$.

14. We shall now subdivide the domain of x for functions of order 2. It is evidently necessary to consider only *Types II* and *III*.

(*a*) Suppose $f(x)$ is of *Type II*. Let c be the point of discontinuity, x_1 and x_1' the points for which $f(x) = x$, so that $x_1 < c < x_1'$.

The domain of x is divided as desired by the points x_1 and x_1'. As x increases from x_1 to x_1', $f(x)$ decreases monotonically from x_1, and passing through $-\infty$ and $+\infty$, decreases again to x_1'.

(*b*) When $f(x)$ is of *Type III*, the domain of x is divided as desired by the point for which $f(x) = x$.

15. We return now to our original problem; that is, given two functions, $f(x)$ and $F(x)$, both of order n, we shall see when there exists a function $\varphi(x)$, which, together with its inverse, satisfies the first two assumptions of § 3, and such that

$$F(x) = \varphi^{-1}f\varphi(x).$$

It will be necessary to consider separately the case of $n = 2$.

Thus, taking $n > 2$, we suppose first that $F(x)$ and $f(x)$ have the same scale m.

Take

$$\varphi F^{pm}(\infty) = f^{pm}(\infty) \qquad (p = 1, 2, \cdots, n-1),$$

$$\varphi(+\infty) = +\infty, \qquad \varphi(-\infty) = -\infty.$$

Take $\varphi(x)$ as any continuous monotone function in the interval $[F^m(\infty), F^{2m}(\infty)]$. Then, following the method of § 2, we can determine $\varphi(x)$ for all values of x. We will find $\varphi(x)$ to be of *Type IV*.

If $F(x)$ has the scale m and $f(x)$ the scale $n - m$, we can take $\varphi F^{pm}(\infty) = f^{pm}(\infty)$, $\varphi(+\infty) = -\infty$ and $\varphi(-\infty) = +\infty$. Proceeding as above, we can find a function $\varphi(x)$ of *Type III*.

Now the reader will not find it very difficult to show that if $f(x)$ has the scale m and if $\varphi(x)$ belongs to one of the four types of § 6, the scale of $F(x) = \varphi^{-1}f\varphi(x)$ will be m or $n - m$ according as φ is increasing or decreasing. That is, neglecting multiples of n, the square of the scale is invariant under the $\varphi^{-1} - \varphi$ transformation.* It is impossible, then, to construct $\varphi(x)$ as desired except in the two cases above. Thus Babbage's formula lacks generality unless we give more liberty to $\varphi(x)$.

---

* To this numerical invariant corresponds an algebraic invariant of all linear fractional functions if we take $\varphi(x)$ linear fractional.

However, in the general case, where $F(x)$ is of scale m and $f(x)$ of scale m', we can, by § 11, determine the integer p, prime to n, so that $f^p(x)$ is of scale m. Then we have

$$F(x) = \varphi^{-1}f^p\varphi(x),$$

where $\varphi(x)$ is of *Type IV*. This formula, then, will give all solutions of Babbage's equation which satisfy the assumptions of § 3, in terms of one such solution. Observe that while we get all solutions by restricting $\varphi(x)$ to *Type IV*, we can also use any of the other types. In fact it would be easy to show that we can replace the $\varphi(x)$ of *Type IV* by another function of *Type I*, discontinuous at any desired point. Observe also that $\varphi(x)$, except for its monotone and continuous nature, is arbitrary within the entire interval $[F^m(\infty), F^{2m}(\infty)]$.

If we give to p values which are not prime to n, the order of $F(x)$ will be a divisor of n. It is easily seen that our formula gives all functions whose orders divide n, except perhaps those of *Types II* and *III* of order 2, should n be even. It will be seen that we do not get these.

16. When $n = 2$ it is necessary to take account of the types to which $f(x)$ and $F(x)$ belong.

Case I. $f(x)$ and $F(x)$ both of *Type I.*

In this case the method of § 15 applies and $\varphi(x)$ can be found, continuous everywhere.

Case II. $f(x)$ and $F(x)$ both of *Type II.*

Let x_1 and x_1' be the points for which $F(x) = x$, X_1 and X_1' the points for which $f(x) = x$. Let c be the point of discontinuity of $F(x)$, and C that of $f(x)$.

Take $\varphi(x_1) = X_1$, $\varphi(c) = C$, $\varphi(x_1') = X_1'$. Take $\varphi(x)$ monotone and continuous in (x_1, x_1'). Then $\varphi(x)$ will be monotone and continuous throughout the domain of x.

Case III. $f(x)$ and $F(x)$ both of *Type III.*

Let $F(x_1) = x_1$ and $f(X_1) = X_1$. Take $\varphi(x_1) = X_1$, and $\varphi(x)$ continuous, and monotone increasing to $+\infty$ or monotone decreasing to $-\infty$ in $(x_1, +\infty)$. Then $\varphi(x)$ will be continuous and monotone everywhere.

Case IV. $f(x)$ of *Type I* and $F(x)$ of *Type II.*

Evidently we cannot determine $\varphi(x)$ as desired, for by § 7, $\varphi^{-1}f\varphi(x)$ has the same monotonic character as $f(x)$.

Case V. $f(x)$ of *Type I* and $F(x)$ of *Type III.*

As in *Case IV* we cannot determine $\varphi(x)$ as desired.

Case VI. $f(x)$ of *Type II* and $F(x)$ of *Type III.*

Let x_1 be the point for which $F(x) = x$, X_1 and X_1' the points for which $f(x) = x$. Take $\varphi(x_1) = X_1$ and $\varphi(\pm\infty) = X_1'$ and $\varphi(x)$ monotone and continuous in the interval $(x_1, +\infty)$. Then $\varphi(x)$ will result monotone wherever continuous, with a single discontinuity. As an example, if $f(x) = 1/x$ and $F(x) = -x$ we can take $\varphi(x) = (x-1)/(x+1)$. A little consideration will show that it is impossible to avoid the discontinuity of $\varphi(x)$ and that the principle employed above is the only one which will give $\varphi(x)$ as desired.

Summarizing the results of this section, we see that in the case of $n = 2$, the formula

$$F(x) = \varphi^{-1}f\varphi(x)$$

gives all solutions of *Type I*, and only those, if $f(x)$ is of *Type I*, and gives all solutions of *Types II* and *III*, and only those, if $f(x)$ is either of *Type II* or *Type III*.

17. Let $f(x)$ and $F(x)$ be two differentiable functions of order 2. Let

$$\varphi(x) = \log\frac{d}{dx}f(x) \text{ and } \Phi(x) = \log\frac{d}{dx}F(x).$$

Put

$$y = f(x) \text{ and } x = f(y).$$

Then

$$1 = \frac{d}{dx}f(y) = \frac{d}{dy}f(y)\frac{d}{dx}f(x)$$

or

$$\varphi(y) + \varphi(x) = 0,$$

so that

$$y = f(x) = \varphi^{-1}[-\varphi(x)],$$

or

$$\varphi f\varphi^{-1}(x) = -x.$$

Similarly,

$$\Phi F\Phi^{-1}(x) = -x,$$

so that

$$F(x) = \Phi^{-1}\varphi f\varphi^{-1}\Phi(x),$$

and putting

$$\varphi^{-1}\Phi(x) = \psi(x),$$

we have

$$F(x) = \psi^{-1}f\psi(x).$$

This reduction can be effected,* irrespective of the types of $F(x)$ and $f(x)$, but, as a general rule, $\psi(x)$ and $\psi^{-1}(x)$ will not conform with the first two assumptions of § 3.

Columbia University.

---

[*] If $f(x) = a - x$ we get an identity on differentiating, but as we have, in this case,

$$f(x) - \frac{a}{2} = -\left(x - \frac{a}{2}\right),$$

the reduction can proceed.